Products of Central Collineations

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ABSTRACT

An n by n matrix M over a (commutative) field F is said to be *central* if $M-I$ has rank 1. We say that M is an *involution* if $M^2=I$; if M is also central we call M a *simple involution*. We will prove that any n-by-n matrix M satisfying $\det M=\pm 1$ is the product of $n+2$ or fewer simple involutions. This can be reduced to $n+1$ if F contains no roots of the equation $x^n=(-1)^n$ other than ± 1. Any ordered field is of this kind. Our main result is that if M is any n-by-n nonsingular nonscalar matrix and if $x_i \in F$ such that $x_1 \cdots x_n = \det M$, then there exist central matrices M_i such that $M=M_1 \cdots M_n$ and $x_i=\det M_i$ for $i=1,\dots,n$. We will apply this result to the projective group $\mathrm{PGL}(n,F)$ and to the little projective group $\mathrm{PSL}(n,F)$.

1. INTRODUCTION

In [5, Theorem 8] Heydar Radjavi proved that any matrix in $\mathrm{GL}(n,F)$ with determinant ± 1 is the product of $2n-1$ or fewer simple involutions. He conjectured that $2n-2$ or fewer factors would not suffice in general. In this paper we will show that, on the contrary, no more than $n+2$ factors are required, and only 3 factors are required for $n=2$ (Lemma 9). For each n, this is the least number of factors that suffices in general. If we delete from consideration all scalar multiples of the identity matrix in $\mathrm{GL}(n,F)$, at most $n+1$ factors are required. If F has characteristic 2, at most $n+1$ factors are required.

We will show that any member of $\mathrm{PSL}(n,F)$ is the product of n or fewer simple involutions (Theorem 3), and is also the product of n or fewer transvections (Theorem 4). These factors can be selected so that the intersection of the fixed spaces of any k of them is isomorphic to the space F^{n-k}/F.

For any triangle Δ in the projective plane F^3/F, and any $T \in \mathrm{PGL}(3,F)$, we will give a necessary and sufficient condition for T to equal the product of 3 or fewer central collineations with distinct axes lying on sides of Δ (Proposition 3). In any case, T equals the product of 4 or fewer central collineations with (not necessarily distinct) axes lying on sides of Δ (Proposition 2).

In [3] and [5] the matrix is first reduced to a canonical form, and then the problem is attacked. We do not use any canonical forms, but rather induction on n. The key to our work is Lemma 8, which concerns not only simple involutions and transvections, but dilatations (homologies) in general.

Here we list the main results of this paper.

THEOREM 1. *Suppose that either $n>2$ or $|F|>2$. Let $T\in \mathrm{PGL}(n,F)$ such that $T\neq$ identity, and let $S_1,\dots,S_n$ be any central collineations in* $\mathrm{PGL}(n,F)$. *Then $S_1\cdots S_nT^{-1}\in \mathrm{PSL}(n,F)$ if and only if there exist central collineations $G_1,\dots,G_n$ such that G_i is conjugate to S_i in the group* $\mathrm{PGL}(n,F)$ *for each i, and*

$$T=G_1\cdots G_n.$$

In particular, such G_i must exist if $\mathrm{PGL}(n,F)=\mathrm{PSL}(n,F)$.

THEOREM 2. *Suppose that either $n>2$ or $|F|>2$. Let T and $S_1,\dots,S_n$ be as in Theorem 1. Fix an index j $(1\leqslant j\leqslant n)$. Then there exist central collineations $G_1,\dots,G_n$ such that G_i is conjugate to S_i in the group* $\mathrm{PGL}(n,F)$ *for all $i\neq j$, and*

$$T=G_1\cdots G_n.$$

THEOREM 3. *Let $T\in \mathrm{PGL}(n,F)$ such that $T\neq$* identity. *Then:*

(a) *When* $\operatorname{char}F\neq 2$, *$T\in\mathrm{PSL}(n,F)$ if and only if T equals the product of n simple involutions, such that the intersection of the fixed spaces of any k of these factors is isomorphic to the space F^{n-k}/F.*

(b) *When* $\operatorname{char}F=2$, *$T\in\mathrm{PSL}(n,F)$ if and only if T equals the product of n or fewer simple involutions, such that the intersection of the fixed spaces of any k of these factors is isomorphic to the space F^{n-k}/F.*

THEOREM 4. *Let $T\in\mathrm{PGL}(n,F)$ such that $T\neq$* identity. *Then $T\in$* $\mathrm{PSL}(n,F)$ *if and only if T equals the product of n or fewer transvections in* $\mathrm{PGL}(n,F)$ *such that the intersection of the fixed spaces of any k of these factors is isomorphic to the space F^{n-k}/F.*

The idea in Theorems 1 and 2 is that we can control the structure of G_i by making G_i conjugate (isomorphic) to some preassigned central collineation S_i. Theorem 2 says we can always control the structure of all but one factor, and Theorem 1 tells us when we can control the structure of all the factors.

2. NOTATION AND NOMENCLATURE

Throughout the paper, F is a (commutative) field and n is an integer >1. F^n is the n-dimensional vector space over F, and $\mathrm{GL}(n,F)$ is the multiplicative group of nonsingular n by n matrices over F. $\mathrm{GL}(n,F)$ is called the *general linear group*. By a *scalar* matrix we mean a matrix of the form cI_n, where $c\in F$ and I_n is the identity matrix in $\mathrm{GL}(n,F)$. A matrix which is not scalar is called *nonscalar*. The subgroup of $\mathrm{GL}(n,F)$ composed of all matrices with determinant 1 is written $\mathrm{SL}(n,F)$. It is called the *special linear group*. We also treat a matrix in $\mathrm{GL}(n,F)$ as a (linear) operator on F^n in the obvious way. Functions will always be written on the left.

If $M\in\mathrm{GL}(n,F)$ and $M-I_n$ has rank 1, we say that M is a *central* matrix. A central matrix with determinant 1 is called a *transvection*. (Some people call this an *elation*.) A central matrix with determinant $\neq 1$ is called a *dilatation*. (Some people call this, variously, a *dilation*, *homology* or *homothety*.) We say that M is an *involution* if $M^2=I_n$. An involution which is a central matrix is called a *simple involution*. Equivalently, a simple involution is a central matrix with determinant -1. Any central matrix with determinant c is similar to the matrix

$$\left[\begin{array}{c|ccc} c & 0 & \cdots & 0 \\ \hline 1 & & & \\ 0 & & & \\ \vdots & & I_{n-1} & \\ 0 & & & \end{array}\right].$$

Thus two central matrices are similar if they have the same determinant. By the *fixed space* of a central matrix we mean its 1-eigenspace.

Sometimes we write $M_1\sim M_2$ to mean that M_1 and M_2 are similar matrices. We say that the matrices $A_1,A_2,A_3,\ldots$ are *simultaneously similar* to $B_1,B_2,B_3,\ldots$ if there exists a $V\in\mathrm{GL}(n,F)$ such that $V^{-1}A_iV=B_i$ for all i.

By $E(i,j,c)$ for $i\neq j$ we mean the matrix which is $c\in F$ in the ijth entry, and coincides with I_n in every other entry. It follows that $E(i,j,c)^{-1}=E(i,j,-c)$. By $E(i,j)$ we mean the result of interchanging the ith and jth rows (columns) of I_n. Then $E(i,j)^{-1}=E(i,j)$. By $E'(i,u)$ we mean the result of multiplying the ith row (column) of I_n by a nonzero $u \in F$. Then $E'(i,u)^{-1} = E'(i,u^{-1})$.

By F^n/F we mean the $(n-1)$-dimensional projective space over F. In particular, F^3/F is the projective plane and F^2/F is the projective line over F. Each matrix in $\mathrm{GL}(n,F)$ implements, in the obvious way, a projective collineation of F^n/F. The multiplicative group of all such collineations is called the *projective group*, written $\mathrm{PGL}(n,F)$. The subgroup of $\mathrm{PGL}(n,F)$ implemented by the matrices in $\mathrm{SL}(n,F)$ is called the *little projective group*, written $\mathrm{PSL}(n,F)$. Note that any scalar matrix implements the identity of $\mathrm{PGL}(n,F)$, and two matrices implement the same projective collineation if one is a scalar multiple of the other. For more detail and a discussion of systems of reference points, consult [1].

By a *central collineation* T, we mean a member of $\mathrm{PGL}(n,F)$ that is implemented by a central matrix M in $\mathrm{GL}(n,F)$. We say that T is a *transvection* if M is a transvection matrix. We say that T is a *dilatation* if M is a dilatation matrix, etc. Of course the fixed space of a central collineation in $\mathrm{PGL}(n,F)$ is isomorphic to F^{n-1}/F. We say that $T \in \mathrm{PGL}(n,F)$ is an *involution* if $T^2 = I$, and T is a *simple involution* if T is also a central collineation.

We say that T_1 and T_2 are *conjugate* in the group $\mathrm{PGL}(n,F)$ if there exists some $S \in \mathrm{PGL}(n,F)$ such that $S^{-1}T_1S = T_2$. This fact is sometimes written $T_1 \sim T_2$.

3. DEVELOPMENT

In this section we develop our main results by means of a series of lemmas about matrices. A check mark $\surd$ in a matrix will stand for some appropriate scalar in F.

LEMMA 1. *Let $x \in F$, and let $M = (m_{ij}) \in \mathrm{GL}(n,F)$ such that either $m_{1j} \neq 0$ for some $j > 1$ or $m_{i1} \neq 0$ for some $i > 1$. Then M is similar to a matrix whose* 11th *entry is x.*

Proof. Let $m_{1j} \neq 0$. Then

$$E\left(1,i,(m_{11}-x)m_{ij}^{-1}\right)^{-1} ME\left(j,1,(x-m_{11})m_{1j}^{-1}\right)$$

suffices. Now let $m_{i1} \neq 0$. Then

$$E\left(1,i,(m_{11}-x)m_{i1}^{-1}\right)^{-1} ME\left(1,i,(m_{11}-x)m_{i1}^{-1}\right)$$

suffices. ∎

LEMMA 2. *Let $x \in F$, and let $M=(m_{ij}) \in \mathrm{GL}(n,F)$ such that M is not diagonal. Then M is similar to a matrix whose* 11th *entry is x.*

Proof. By Lemma 1, we can suppose without loss of generality that $m_{1j}=0$ for all $j>1$ and $m_{i1}=0$ for all $i>1$. Let $m_{ij}\neq 0$ for some i,j satisfying $i\neq j$. Then

$$E(1,i,1)^{-1}ME(1,i,1)$$

has $-m_{ij}\neq 0$ in the $1j$th place. But $j>1$, since $m_{ij}\neq 0$. Apply Lemma 1 to this matrix. ■

LEMMA 3. *Let $x\in F$, and let $M=(m_{ij})\in \mathrm{GL}(n,F)$ be nonscalar. Then M is similar to a matrix whose* 11th *entry is x.*

Proof. By Lemma 2 we can assume without loss of generality that M is diagonal. But M is not scalar. So $m_{ii}\neq m_{11}$ for some $i>1$. Then the $1i$th entry of

$$E(1,i,1)^{-1}ME(1,i,1)$$

is $m_{11}-m_{ii}\neq 0$. Apply Lemma 1 to this matrix. ■

LEMMA 4. *Let $M=(m_{ij})\in \mathrm{GL}(2,F)$ be nonscalar, and let $x_1,x_2\in F$ such that $x_1x_2=\det M$. Then M is similar to a matrix product of the form*

$$\begin{pmatrix} x_1 & 0 \\ \vee & 1 \end{pmatrix}\begin{pmatrix} 1 & \vee \\ 0 & x_2 \end{pmatrix}.$$

Proof. By Lemma 3, M is similar to a matrix

$$\begin{pmatrix} x_1 & a \\ b & c \end{pmatrix}=\begin{pmatrix} x_1 & 0 \\ b & 1 \end{pmatrix}\begin{pmatrix} 1 & ax_1^{-1} \\ 0 & c-abx_1^{-1} \end{pmatrix}$$

for appropriate scalars $a,b,c\in F$. Then the determinant of the product on the right is $x_1(c-abx_1^{-1})$, and also $x_1x_2=\det M$. So $x_2=c-abx_1^{-1}$. ■

Our next project is to extend Lemma 4 to matrices in $\mathrm{GL}(n,F)$ for $n\geqslant 2$ by induction on n.

LEMMA 5. *Let* $B_2,\dots,B_n \in \mathrm{GL}(n-1,F)$ *where* $n \geqslant 3$ *and*

$$B_2=\left[\begin{array}{c|ccc} a_{22} & 0 & \cdots & 0 \\ \hline \surd & & & \\ \vdots & & I_{n-2} & \\ \surd & & & \end{array}\right], \qquad B_3=\left[\begin{array}{cc|ccc} 1 & a_{23} & 0 & \cdots & 0 \\ 0 & a_{33} & 0 & \cdots & 0 \\ \hline 0 & \surd & & & \\ \vdots & \vdots & & I_{n-3} & \\ 0 & \surd & & & \end{array}\right],$$

$$B_4=\left[\begin{array}{ccc|ccc} 1 & 0 & a_{24} & 0 & \cdots & 0 \\ 0 & 1 & a_{34} & 0 & \cdots & 0 \\ 0 & 0 & a_{44} & 0 & \cdots & 0 \\ \hline 0 & 0 & \surd & & & \\ \vdots & \vdots & \vdots & & I_{n-4} & \\ 0 & 0 & \surd & & & \end{array}\right],\dots, \qquad B_n=\left[\begin{array}{ccc|c} & & & a_{2n} \\ & & & a_{3n} \\ & I_{n-1} & & \vdots \\ \hline 0 & \cdots & 0 & a_{nn} \end{array}\right].$$

Let $a_{12},\dots,a_{1n} \in F$, *and let* $B_2 \cdots B_n = B$. *Then there exist unique scalars* $u_2,\dots,u_n \in F$ *such that*

$$\left[\begin{array}{c|cccc} 1 & u_2 & 0 & \cdots & 0 \\ \hline 0 & & & & \\ \vdots & & & B_2 & \\ 0 & & & & \end{array}\right]\left[\begin{array}{c|ccccc} 1 & 0 & u_3 & 0 & \cdots & 0 \\ \hline 0 & & & & & \\ \vdots & & & B_3 & & \\ 0 & & & & & \end{array}\right]\cdots$$

$$\times\left[\begin{array}{c|cccc} 1 & 0 & \cdots & 0 & u_n \\ \hline 0 & & & & \\ \vdots & & & B_n & \\ 0 & & & & \end{array}\right]=\left[\begin{array}{c|cccc} 1 & a_{12} & a_{13} & \cdots & a_{1n} \\ \hline 0 & & & & \\ \vdots & & & B & \\ 0 & & & & \end{array}\right].$$

Proof. What is required is the equations

$$u_2 = a_{12}, \qquad u_3 + u_2 a_{23} = a_{13},$$
$$u_4 + u_2 a_{24} + (u_3 + u_2 a_{23}) a_{34} = a_{14}, \qquad \cdots$$

obtained from the top row. Note that the kth equation is of the form

$$u_{k+1} + (\text{terms in } u_2, \dots, u_k \text{ and } a_{ij}) = a_{1,k+1},$$

so we can solve consecutively for $u_2, u_3, u_4, \dots$. ■

LEMMA 6. *Let $M \in \mathrm{GL}(n,F)$ be nonscalar, $n \geqslant 3$, $x \in F$, $x \neq 0$. Then M is similar to a matrix whose first row is the vector $(x, x, 0, \dots, 0)$.*

Proof. We first claim that M is similar to a matrix whose first row is of the form $(x, \vee, \dots, \vee)$, but not $(x, 0, \dots, 0)$. By Lemma 3, it suffices to suppose, without loss of generality, that the first row of M is in fact $(x, 0, \dots, 0)$. Indeed it suffices to suppose that all the diagonal elements of $M = (m_{ij})$ are x; for if $m_{ii} \neq x$ $(i > 1)$, then

$$E(1,i,1)^{-1} E\left(i, 1, m_{i1}(x - m_{ii})^{-1}\right)^{-1} M E\left(i, 1, m_{i1}(x - m_{ii})^{-1}\right) E(1,i,1)$$

has x in the 11th place and $x - m_{ii} \neq 0$ in the 1ith place. But M is not scalar, so for some i, j satisfying $i \neq j$, we have $m_{ij} \neq 0$. Then $i > 1$, and

$$E(i,1)^{-1} M E(i,1)$$

has x in the 11th place and $m_{ij} \neq 0$ in the first row. This proves our claim.

So we can suppose without loss of generality that $m_{11} = x$ and $m_{1j} \neq 0$ for some $j > 1$. But

$$E(j,2)^{-1} M E(j,2)$$

has x in the 11th place and is nonzero in the 12th place. So suppose $m_{11} = x$ and $m_{12} \neq 0$. Then the first row of the matrix

$$E\left(2, n, -m_{1n} m_{12}^{-1}\right)^{-1} \cdots E\left(2, 3, -m_{13} m_{12}^{-1}\right)^{-1}$$

$$\times\, M E\left(2, 3, -m_{13} m_{12}^{-1}\right) \cdots E\left(2, n, -m_{1n} m_{12}^{-1}\right)$$

is $(x, m_{12}, 0, \dots, 0)$, and we again assume M is this matrix. But the first row of the matrix

$$E'\left(2,m_{12}^{-1}x\right)^{-1}ME'\left(2,m_{12}^{-1}x\right)$$

is $(x,x,0,\dots,0)$. ∎

LEMMA 7. *Let $M\in\mathrm{GL}(n,F)$ be nonscalar, $n\geqslant 3$, $x\in F$, $x\neq 0$. Then M is similar to a matrix product of the form*

$$\left[\begin{array}{c|ccc} x & 0 & \cdots & 0 \\ \hline \vee & & & \\ \vdots & & I_{n-1} & \\ \vee & & & \end{array}\right]\left[\begin{array}{c|ccc} 1 & \vee & \cdots & \vee \\ \hline 0 & & & \\ \vdots & & B & \\ 0 & & & \end{array}\right],$$

where $B\in\mathrm{GL}(n-1,F)$ is nonscalar.

Proof. By Lemma 6, M is similar to a matrix

$$Q=\begin{bmatrix} x & x & 0 & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

$$=\left[\begin{array}{c|ccc} x & 0 & \cdots & 0 \\ \hline a_{21} & & & \\ \vdots & & I_{n-1} & \\ a_{n1} & & & \end{array}\right]\left[\begin{array}{c|cccc} 1 & 1 & 0 & \cdots & 0 \\ \hline 0 & & & & \\ \vdots & & & C & \\ 0 & & & & \end{array}\right]$$

for some $C=(c_{ij})\in\mathrm{GL}(n-1,F)$. Just put $c_{i-1,1}=a_{i2}-a_{i1}$ and $c_{i-1,j-1}=a_{ij}$ for $i>1$, $j>2$. Without loss of generality, we suppose that C is a scalar matrix cI_{n-1}. Then M is similar to the product

$$E(3,1,1)^{-1}\left[\begin{array}{c|ccc} x & 0 & \cdots & 0 \\ \hline a_{21} & & & \\ \vdots & & I_{n-1} & \\ a_{n1} & & & \end{array}\right]E(3,1,1)E(3,1,1)^{-1}\ \times$$

$$\times \left[\begin{array}{c|cccc} 1 & 1 & 0 & \cdots & 0 \\ \hline 0 & & & & \\ \vdots & & C & & \\ 0 & & & & \end{array}\right] E(3,1,1)$$

$$= \left[\begin{array}{c|ccc} x & 0 & \cdots & 0 \\ \hline \surd & & & \\ \vdots & & I_{n-1} & \\ \surd & & & \\ \surd & & & \end{array}\right] \left[\begin{array}{cccccc} 1 & 1 & 0 & \cdots & & 0 \\ 0 & c & 0 & \cdots & & 0 \\ c-1 & -1 & c & & & \\ 0 & 0 & 0 & & & 0 \\ \vdots & & & & \ddots & \\ 0 & & 0 & & & c \end{array}\right]$$

$$= \left[\begin{array}{c|ccc} x & 0 & \cdots & 0 \\ \hline \surd & & & \\ \vdots & & I_{n-1} & \\ \surd & & & \\ \surd & & & \end{array}\right] \left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ c-1 & & & \\ 0 & & I_{n-1} & \\ \vdots & & & \\ 0 & & & \end{array}\right] \left[\begin{array}{ccccccc} 1 & 1 & 0 & 0 & \cdots & & 0 \\ 0 & c & 0 & 0 & \cdots & & 0 \\ 0 & -c & c & 0 & \cdots & & 0 \\ 0 & 0 & 0 & c & & & 0 \\ \vdots & \vdots & \vdots & & & \ddots & \\ 0 & 0 & 0 & 0 & & & c \end{array}\right]$$

$$= \left[\begin{array}{c|ccc} x & 0 & \cdots & 0 \\ \hline \surd & & & \\ \vdots & & I_{n-1} & \\ \surd & & & \\ \surd & & & \end{array}\right] \left[\begin{array}{ccccccc} 1 & 1 & 0 & 0 & \cdots & & 0 \\ 0 & c & 0 & 0 & \cdots & & 0 \\ 0 & -c & c & 0 & \cdots & & 0 \\ 0 & 0 & 0 & c & & & 0 \\ \vdots & \vdots & \vdots & & & \ddots & \\ 0 & 0 & 0 & 0 & & & c \end{array}\right].$$

Finally $c \neq 0$, since the last matrix is nonsingular, so the matrix

$$\begin{bmatrix} c & 0 & 0 & \cdots & & 0 \\ -c & c & 0 & \cdots & & 0 \\ 0 & 0 & c & & & 0 \\ \vdots & \vdots & & & \ddots & \\ 0 & 0 & 0 & & & c \end{bmatrix}$$

is nonscalar. ■

Now we are ready to generalize Lemma 4 to $\mathrm{GL}(n,F)$.

LEMMA 8. *Let $M=(m_{ij})\in \mathrm{GL}(n,F)$ be nonscalar, and let $x_1,\ldots,x_n\in F$ be scalars such that $x_1\cdots x_n=\det M$. Then M is similar to some matrix product of the form*

$$\left[\begin{array}{c|ccc} x_1 & 0 & \cdots & 0 \\ \hline \checkmark & & & \\ \vdots & & I_{n-1} & \\ \checkmark & & & \\ \checkmark & & & \end{array}\right]\left[\begin{array}{cc|ccc} 1 & \checkmark & 0 & \cdots & 0 \\ 0 & x_2 & 0 & \cdots & 0 \\ 0 & \checkmark & & & \\ \cline{3-5} \vdots & \vdots & & I_{n-2} & \\ 0 & \checkmark & & & \\ 0 & \checkmark & & & \end{array}\right]\cdots\left[\begin{array}{ccc|c} & & & \checkmark \\ & I_{n-1} & & \checkmark \\ & & & \vdots \\ & & & \checkmark \\ \hline 0 & \cdots & 0 & x_n \end{array}\right],$$

where each check mark $\checkmark$ stands for an appropriate scalar.

Proof is by induction on n. For $n=2$, this is Lemma 4. Assume that the result is true for $\mathrm{GL}(k,F)$, $k=2,\ldots,n-1$. By Lemma 7, M is similar to a matrix product of the form

$$Q=\left[\begin{array}{c|ccc} x_1 & 0 & \cdots & 0 \\ \hline \checkmark & & & \\ \vdots & & I_{n-1} & \\ \checkmark & & & \end{array}\right]\left[\begin{array}{c|ccc} 1 & \checkmark & \cdots & \checkmark \\ \hline 0 & & & \\ \vdots & & B & \\ 0 & & & \end{array}\right],$$

where $B\in \mathrm{GL}(n-1,F)$ is nonscalar. The left factor has determinant x_1, and $\det Q=x_1x_2\cdots x_n$. So the right factor has determinant $x_2\cdots x_n$, and $\det B=x_2\cdots x_n$. By the induction hypothesis, B can be factored

$$V^{-1}BV=\left[\begin{array}{c|ccc} x_2 & 0 & \cdots & 0 \\ \hline \checkmark & & & \\ \vdots & & I_{n-2} & \\ \checkmark & & & \end{array}\right]\left[\begin{array}{cc|ccc} 1 & \checkmark & 0 & \cdots & 0 \\ 0 & x_3 & 0 & \cdots & 0 \\ \hline 0 & \checkmark & & & \\ \vdots & \vdots & & I_{n-3} & \\ 0 & \checkmark & & & \end{array}\right]\times$$

$$\cdots\left[\begin{array}{ccc|c} & & & \checkmark \\ & I_{n-2} & & \vdots \\ & & & \checkmark \\ \hline 0 & \cdots & 0 & x_n \end{array}\right] \tag{1}$$

for some $V \in \mathrm{GL}(n-1,F)$. Also

$$W^{-1}QW = W^{-1}\left[\begin{array}{c|ccc} x_1 & 0 & \cdots & 0 \\ \hline \surd & & & \\ \vdots & & I_{n-1} & \\ \surd & & & \end{array}\right] WW^{-1}\left[\begin{array}{c|ccc} 1 & \surd & \cdots & \surd \\ \hline 0 & & & \\ \vdots & & B & \\ 0 & & & \end{array}\right] W,$$

where

$$W = \left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & V & \\ 0 & & & \end{array}\right],$$

and

$$W^{-1}QW = \left[\begin{array}{c|ccc} x_1 & 0 & \cdots & 0 \\ \hline \surd & & & \\ \vdots & & I_{n-1} & \\ \surd & & & \end{array}\right]\left[\begin{array}{c|ccc} 1 & \surd & \cdots & \surd \\ \hline 0 & & & \\ \vdots & & V^{-1}BV & \\ 0 & & & \end{array}\right]. \tag{2}$$

Let A_{k+1} denote the kth factor in (1). By Lemma 5, there exist scalars $u_2, \ldots, u_n$ such that

$$\left[\begin{array}{c|ccc} 1 & \surd & \cdots & \surd \\ \hline 0 & & & \\ \vdots & & V^{-1}BV & \\ 0 & & & \end{array}\right] = \left[\begin{array}{c|cccc} 1 & u_2 & 0 & \cdots & 0 \\ \hline 0 & & & & \\ \vdots & & A_2 & & \\ 0 & & & & \end{array}\right]$$

$$\times \left[\begin{array}{c|ccccc} 1 & 0 & u_3 & 0 & \cdots & 0 \\ \hline 0 & & & & & \\ \vdots & & & A_3 & & \\ 0 & & & & & \end{array}\right] \cdots \left[\begin{array}{c|cccc} 1 & 0 & \cdots & 0 & u_n \\ \hline 0 & & & & \\ \vdots & & A_n & & \\ 0 & & & & \end{array}\right]. \tag{3}$$

We combine (2) and (3) to see that Q and M are similar to the desired matrix product. This completes the induction. ∎

Before we tackle the proofs of our theorems, observe that Lemma 8 fails when M is a scalar matrix. Consider $M = I_2$, $x_1 = 2$, $x_2 = \frac{1}{2}$. One reason Lemma 8 does not work for scalar matrices is that such a matrix is similar to no matrix but itself. On the other hand, Lemma 8 suggests that any nonscalar matrix M is similar to several other matrices.

Let

$$V^{-1}MV=A_1\cdots A_n$$

be the equation in $\mathrm{GL}(n,F)$ given by Lemma 8. Let u be any nonzero scalar in F. Then

$$[VE'(1,u)]^{-1}M[VE'(1,u)]=E'(1,u)^{-1}A_1E'(1,u)\cdots E'(1,u)^{-1}A_nE'(1,u),$$

and each $E'(1,u)^{-1}A_iE'(1,u)$ satisfies the requirements of Lemma 8, because A_i does. But $\det[VE'(1,u)]=u\det V$. In other words, we can control the determinant of the matrix V implementing the similarity as well as the determinants of the factors A_i.

Proof of Theorem 1. Suppose the central collineations $G_1,\dots,G_n$ exist as stated; $T=G_1\cdots G_n$. Thus there is a system of reference points for F^n/F such that $M(T)=M(G_1)\cdots M(G_n)$, where $M(G_i)$ is a matrix implementing G_i relative to this system, etc. Since G_i is conjugate to S_i in $\mathrm{PGL}(n,F)$, $M(S_i)$ can be made similar to $M(G_i)$, and $\det M(S_i)=\det M(G_i)$. But $\det M(T)=\det M(G_1)\cdots\det M(G_n)=\det M(S_1)\cdots M(S_n)$. Then

$$\det M[S_1\cdots S_nT^{-1}]=\det M(S_1)\cdots\det M(S_n)\det{}^{-1}M(T)=1,$$

and $S_1\cdots S_nT^{-1}\in\mathrm{PSL}(n,F)$.

Now suppose that $S_1\cdots S_nT^{-1}\in\mathrm{PSL}(n,F)$. Take a system of reference points for F^n/F, and choose $M(S_i)$, $M(T)$ such that

$$\det M(S_1)\cdots\det M(S_n)=\det M(T),$$

and each $M(S_i)$ is a central matrix. Put $x_i=\det M(S_i)$ and find the matrices $M(G_1),\dots,M(G_n)$ given by Lemma 8; $\det M(G_i)=x_i=\det M(S_i)$ for all i, and $M(T)\sim M(G_1)\cdots M(G_n)$. We can suppose, without loss of generality, that none of the factors $M(G_i)$ is the identity matrix. Note that

$$I_n\left[\begin{array}{c|ccc} x & 0 & \cdots & 0\\ \hline u_2 & & & \\ \vdots & & I_{n-1} & \\ u_n & & & \end{array}\right]=\left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0\\ \hline y_2 & & & \\ \vdots & & I_{n-1} & \\ y_n & & & \end{array}\right]\left[\begin{array}{c|ccc} x & 0 & \cdots & 0\\ \hline u_2-xy_2 & & & \\ \vdots & & I_{n-1} & \\ u_n-xy_n & & & \end{array}\right],$$

$$\left[\begin{array}{c|ccc} x & 0 & \cdots & 0\\ \hline u_2 & & & \\ \vdots & & I_{n-1} & \\ u_n & & & \end{array}\right]I_n=\left[\begin{array}{c|ccc} x & 0 & \cdots & 0\\ \hline u_2-y_2 & & & \\ \vdots & & I_{n-1} & \\ u_n-y_n & & & \end{array}\right]\left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0\\ \hline y_2 & & & \\ \vdots & & I_{n-1} & \\ y_n & & & \end{array}\right],$$

so I_n and

$$\left[\begin{array}{c|ccc} x & 0 & \cdots & 0 \\ \hline u_2 & & & \\ \vdots & & I_{n-1} & \\ u_n & & & \end{array}\right]$$

can be replaced by central matrices $\neq I_n$, since $n=|F|=2$ does not hold, etc. It follows that G_i is conjugate to S_i in $\mathrm{PGL}(n,F)$, and $T \sim G_1 \cdots G_n$. ∎

We remark that Theorem 1 does not hold for $n=|F|=2$. In this case $\mathrm{PGL}(2,F)$, $\mathrm{PSL}(2,F)$, $\mathrm{GL}(2,F)$, $\mathrm{SL}(2,F)$ are all the non-Abelian group of order 6, and

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is not the product of two central matrices.

Moreover, Theorem 1 is false for $T=$ identity. Let $n=2$, and let S_1 be implemented by the matrix $\mathrm{diag}(8,1)$, and S_2 be implemented by the matrix $\mathrm{diag}(2,1)$. Then $S_1S_2 \in \mathrm{PSL}(2,F)$, since $4^2=16$. Suppose G_i is conjugate to S_i in $\mathrm{PGL}(2,F)$ and $G_1G_2=$ identity. There must be matrices A_1 similar to $\mathrm{diag}(8,1)$, A_2 similar to $\mathrm{diag}(2,1)$, such that $A_1A_2=\mathrm{diag}(4,4)$. Indeed, there is a $V \in \mathrm{GL}(2,F)$ such that $\mathrm{diag}(8,1)\,V^{-1}A_2V=\mathrm{diag}(4,4)$. So $V^{-1}A_2V=\mathrm{diag}(\frac{1}{2},4)$, which is impossible, since the eigenvalues of A_2 are not $\frac{1}{2}$ and 4.

Proof of Theorem 2. Take some reference system of F^n/F and the central matrices $M(S_i)$ $(i=1,\dots,n)$ and $M(T)$. Put $x_i=\det M(S_i)$ for $i \neq j$, and choose x_j so that $x_1 \cdots x_n=\det M(T)$. The rest of the proof parallels the proof of Theorem 1, and the details are straightforward. ∎

A point worth mentioning is that we can make G_i conjugate to S_i for all i in Theorem 2 if $\mathrm{PGL}(n,F)=\mathrm{PSL}(n,F)$ (equivalently, if every element in F has an nth root in F). This follows from Theorem 1.

If $T \in \mathrm{PGL}(n,F)$ is the product of n simple involutions, then T is implemented by a matrix with determinant $(-1)^n$, and so $T \in \mathrm{PSL}(n,F)$. In particular, for $\mathrm{char}\,F=2$ any product of simple involutions must lie in $\mathrm{PSL}(n,F)$.

Proof of Theorem 3.

(a) Choose a reference system for F^n/F, and let $M(T)$ be a matrix implementing T. Suppose $T \in \mathrm{PSL}(n,F)$. Then we can select $M(T)$ so that

$\det M(T)=(-1)^n$. Let $x_1=\cdots=x_n=-1$ in Lemma 8, and let $M(G_1),\dots,M(G_n)$ be the matrices given by Lemma 8; $M(T)\sim M(G_1)\cdots M(G_n)$. None of the $M(G_i)$ can be scalar, because they have both 1s and -1s on their main diagonals. Thus each G_i is a simple involution, and $T\sim G_1\cdots G_n$. The converse is straightforward.

(b) Suppose $T\in\mathrm{PSL}(n,F)$. Let $x_1=\cdots=x_n=1$ in Lemma 8, and let $M(G_1),\dots,M(G_n)$ be the matrices given by Lemma 8; $M(T)\sim M(G_1)\cdots M(G_n)$. Then $T\sim\prod_i G_i$, where we delete all the factors G_i equal to the identity. The converse is straightforward. ∎

Proof of Theorem 4. Suppose $T\in\mathrm{PSL}(n,F)$. Choose a reference system, and select $M(T)$ so that $\det M(T)=1$. Let $x_1=\cdots=x_n=1$ in Lemma 8, and let $M(G_1),\dots,M(G_n)$ be the matrices given by Lemma 8; $M(T)\sim M(G_1)\cdots M(G_n)$. Then $T\sim\prod_i G_i$, where we delete all factors G_i that equal the identity. Of course, all the remaining factors are transvections. The converse is straightforward. ∎

If $T\in\mathrm{PGL}(n,F)$ and $T^n\neq$ identity, then $T^n\in\mathrm{PSL}(n,F)$, so that T^n can be expressed as a product of simple involutions as in Theorem 3 and as the product of transvections as in Theorem 4.

If $\mathrm{PGL}(3,F)=\mathrm{PSL}(3,F)$ and if $T\in\mathrm{PGL}(3,F)$, $T\neq$ identity, then T can be expressed as the product of 3 central collineations in any one of the 8 patterns formed by making each factor a transvection or a homology. This can be extended in an obvious manner to $\mathrm{PGL}(n,F)$ for $n\geqslant 2$, in which the pattern of the n factors can be preassigned.

Now we need another matrix lemma.

LEMMA 9. *Let M be a nonscalar member of* $\mathrm{GL}(n,F)$ *and $u\in F$. Then*

(a) *if* $\det M=(-1)^n$, *M is the product of n or fewer simple involutions,*

(b) *if* $\det M=(-1)^{n+1}$, *M is the product of $n+1$ or fewer simple involutions,*

(c) *if $u^n=(-1)^{n+1}$, uI_n is the product of $n+1$ or fewer simple involutions,*

(d) *if $u^n=(-1)^n$, uI_n is the product of $n+2$ or fewer simple involutions.*

Proof.

(a) Clear by Lemma 8.

(b) Use $x_1=1$, $x_2=\cdots=x_n=-1$ in Lemma 8. Replace the first factor B with $E'(1,-1)[E'(1,-1)B]$.

(c) Let A be the matrix which is $\begin{pmatrix}0&1\\1&0\end{pmatrix}$ in the first 2 rows and first 2 columns, and which coincides with I_n in every other entry. Then A is clearly a simple involution and $\det[A(uI_n)] = -u^n = (-1)^n$. Also $A(uI_n)$ is nonscalar. Apply Lemma 8 to $A(uI_n)$.

(d) This time $\det[A(uI_n)] = -u^n = (-1)^{n+1}$. Apply part (b) to $A(uI_n)$. ∎

For $n \geqslant 3$, we show that the numbers of factors given in (a), (b), (c), (d) are the smallest possible. Let F be the field of complex numbers. If in case (a), (b) or (c) fewer factors will suffice, then fewer than n factors will suffice. Thus 1 is an eigenvalue of M or of uI_n. But we can make M a diagonal matrix such that $M - I_n \in \mathrm{GL}(n,F)$, and we can make u a primitive nth root of $(-1)^{n+1}$.

Case (d) is harder. Let u be a primitive nth root of $(-1)^n$. Suppose that fewer than $n+2$ factors suffice. Then fewer than $n+1$ factors suffice. Since 1 is not an eigenvalue of uI_n, exactly n factors are required. Say $uI_n = A_1 \cdots A_n$, where each A_i is a simple involution. Since 1 is not an eigenvalue of uI_n, the 1-eigenspace of each A_i does not contain the intersection of the 1-eigenspaces of all the A_j ($j \neq i$). For each $i = 1, \dots, n$, choose a vector z_i in the intersection of all the 1-eigenspaces of the A_j ($j \neq i$), such that z_i is not in the 1-eigenspace of A_i. It follows that $\{z_1, \dots, z_n\}$ is a linearly independent subset of F^n, so $\{z_1, \dots, z_n\}$ is an ordered basis of F^n. There exists a change-of-basis matrix $V \in \mathrm{GL}(n,F)$ such that the matrices $B_i = V^{-1}A_iV$ have the form of the factors in Lemma 8. Moreover,

$$B_1 \cdots B_n = V^{-1}(A_1 \cdots A_n)V = uI_n.$$

But $B_1 \cdots B_n$ has -1 in the 11th place, and uI_n has u in the 11th place. Impossible!

For each $n \geqslant 2$, let $f(n)$ denote the minimal number of factors required to express every matrix in $\mathrm{GL}(n,F)$ with determinant ± 1 as a product of simple involutions in $\mathrm{GL}(n,F)$. We have shown that $f(n) = n+2$ for $n \geqslant 3$. It is easy to see that $f(2) = 3$. This answers a question raised by H. Radjavi [5] and shows that his conjecture is false.

Though Lemma 8 does not apply to scalar matrices M, we do have a reasonable analogue for scalar matrices when an extra factor is allowed. Let $M \in \mathrm{GL}(n,F)$ be a scalar matrix, and let $x_1, \dots, x_{n+1} \in F$ such that $x_1 \cdots x_{n+1} = \det M$. Then there exist central matrices $A_1, \dots, A_{n+1}$ such that $M = A_1 \cdots A_{n+1}$ and $\det A_i = x_i$ ($i = 1, \dots, n+1$). To prove this, let B denote the matrix in $\mathrm{GL}(n,F)$ that is

$$\begin{pmatrix} x_1^{-1} & 0 \\ 1 & 1 \end{pmatrix}$$

in the first 2 rows and first 2 columns, and that coincides with I_n in every other entry. Then B is central, $\det B = x_1^{-1}$, and BM is nonscalar. Also $\det(BM) = x_2 \cdots x_{n+1}$. Apply Lemma 8 to the matrix BM, etc.

Of course, some involutions are not simple—for example, $\operatorname{diag}(1,1,-1,-1) \in \mathrm{GL}(4,F)$ when $\operatorname{char} F \neq 2$. Gustafson, Halmos and Radjavi [3] have shown that any $M \in \mathrm{GL}(n,F)$ satisfying $\det M = \pm 1$ is the product of 4 or fewer involutions. From this it easily follows that any member of $\mathrm{PSL}(n,F)$ equals the product of 4 or fewer involutions in $\mathrm{PGL}(n,F)$.

We turn now to some specific results about the projective plane F^3/F. To this end, we consider $\mathrm{GL}(3,F)$ and $\mathrm{PGL}(3,F)$. This portion of our work was inspired by an incorrect proof [6, pp. 130–131] of the theorem that any member of $\mathrm{PGL}(3,F)$ equals the product of 3 or fewer central collineations in $\mathrm{PGL}(3,F)$.

Lemma 10. *Let*

$$M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

be a matrix such that no 2 of its rows are linearly dependent vectors in F^3. Let

$$M_1 = \begin{bmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} 1 & y_1 & 0 \\ 0 & y_2 & 0 \\ 0 & y_3 & 1 \end{bmatrix}, \qquad M_3 = \begin{bmatrix} 1 & 0 & z_1 \\ 0 & 1 & z_2 \\ 0 & 0 & z_3 \end{bmatrix}.$$

Then there exists a solution in $x_i, y_i, z_i \in F$ to the equation $M = M_1 M_2 M_3$ if and only if $a^2 e \neq abd$; and if a solution exists, it is unique. Likewise the corresponding statement holds for

(1) $M = M_2 M_1 M_3$ *and* $ae^2 \neq bde$,
(2) $M = M_3 M_2 M_1$ *and* $ei^2 \neq fhi$,
(3) $M = M_1 M_3 M_2$ *and* $a^2 i \neq acg$,
(4) $M = M_3 M_1 M_2$ *and* $ai^2 \neq cgi$,
(5) $M = M_2 M_3 M_1$ *and* $e^2 i \neq efh$.

Proof.

$$M_1 M_2 M_3 = \begin{bmatrix} x_1 & x_1 y_1 & x_1 z_1 + x_1 y_1 z_2 \\ x_2 & x_2 y_1 + y_2 & x_2 z_1 + (x_2 y_1 + y_2) z_2 \\ x_3 & x_3 y_1 + y_3 & x_3 z_1 + (x_3 y_1 + y_3) z_2 + z_3 \end{bmatrix}.$$

Thus $M=M_1M_2M_3$ is equivalent to all these equations holding simultaneously: $x_1=a$, $x_2=d$, $x_3=g$, $ay_1=b$, $ay_2=ae-bd$, $ay_3=ah-gb$, $(ae-bd)z_1=ce-fb$, $(ae-bd)z_2=af-cd$, $z_3=i-(gz_1+hz_2)$.

Suppose that $a^2e\neq abd$. Then $a\neq 0$ and $ae-bd\neq 0$, and clearly $M=M_1M_2M_3$ has a unique solution.

Now suppose that $M=M_1M_2M_3$ has a solution. Then $ae-bd\neq 0$; for if $ae-bd=0$, then we see from the above equations that $ce-fb=af-cd=0$, and any matrix whose first two rows coincide with those of M is singular (just compute the determinant to get this), and hence the first two rows of M are linearly dependent in F^3, contrary to hypothesis. Also $a\neq 0$; for if $a=0$, it follows that $ae-bd=0$, which we have seen is impossible. Thus

$$a(ae-bd)=a^2e-abd\neq 0.$$

For (1), note that M, M_2, M_1, M_3 are simultaneously similar to

$$\begin{bmatrix} e & d & f \\ b & a & c \\ h & g & i \end{bmatrix}, \quad M_1', \quad M_2', \quad M_3'.$$

Use

$$V=\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $M\to V^{-1}MV$, etc.

For (2), note that M, M_3, M_2, M_1 are simultaneously similar to

$$\begin{bmatrix} i & h & g \\ f & e & d \\ c & b & a \end{bmatrix}, \quad M_1', \quad M_2', \quad M_3'.$$

Use

$$V=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

For (3), note that M, M_1, M_3, M_2 are simultaneously similar to

$$\begin{bmatrix} a & c & b \\ g & i & h \\ d & f & e \end{bmatrix}, \quad M_1', \quad M_2', \quad M_3'.$$

Use

$$V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

For (4), note that M, M_3, M_1, M_2 are simultaneously similar to

$$\begin{bmatrix} i & g & h \\ c & a & b \\ f & d & e \end{bmatrix}, \quad M_1', \quad M_2', \quad M_3'.$$

Use

$$V = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

For (5), note that M, M_2, M_3, M_1 are simultaneously similar to

$$\begin{bmatrix} e & f & d \\ h & i & g \\ b & c & a \end{bmatrix}, \quad M_1', \quad M_2', \quad M_3'.$$

Use

$$V = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The conclusion follows. ■

For example,

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

is a nonsingular matrix which is not equal to the product of 3 matrices of the forms M_1, M_2, M_3 in any order. Of course, it is similar to such a matrix product.

It is interesting that though there exists a matrix which is not the product of 3 matrices of the forms M_i of Lemma 10, any 3-by-3 matrix is the product of 4 or fewer such matrices, as we shall now see. Here we do not claim that our product is unique.

PROPOSITION 1. *Let M be any 3-by-3 matrix. Then M equals the product of 4 or fewer matrices of the forms M_1, M_2, M_3 given in Lemma* 10, *in some order.*

Proof. We dispose of the zero matrix right away:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now suppose that $\operatorname{rank} M = 1$. Without loss of generality we can replace M with the matrix $V^{-1}MV$, where V is one of the 5 permutation matrices used in the proof of Lemma 10. Thus we may suppose, without loss of generality, that there is a nonzero entry in the first column of M. Then M can be expressed as

$$M = \begin{pmatrix} a & ua & va \\ b & ub & vb \\ c & uc & vc \end{pmatrix} = \begin{bmatrix} a & 0 & 0 \\ b & 1 & 0 \\ c & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & u & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & v \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Finally, we suppose that $\operatorname{rank} M > 1$. Then 2 columns of M are linearly independent in F^3. By replacing M with $V^{-1}MV$, where V is an appropriate permutation matrix of the kind used in the proof of Lemma 10, we suppose without loss of generality that the first 2 columns of M are linearly independent vectors in F^3. Let

$$M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

The rest of the proof is divided into cases.

Case 1. $ae \neq bd$ and either $a \neq 0$ or $e \neq 0$. Proceed as in the proof of Lemma 10. Only 3 factors are needed.

Case 2. $a=e=0$, and $b\neq 0$, $d\neq 0$. Apply case 1 to the matrix

$$\begin{bmatrix} 0 & b & c \\ d & 0 & f \\ g & h & i \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} 0 & b & c \\ d & d & f \\ g & g+h & i \end{bmatrix}.$$

Thus we have proved the conclusion for $ae\neq bd$.

Case 3. $ae=bd$ and either $dh\neq eg$ or $ah\neq bg$. Routine calculations show that for appropriate scalars $u,v\in F$, the hypothesis of case 1 is satisfied by the matrix

$$\begin{bmatrix} 1 & 0 & u \\ 0 & 1 & v \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}=\begin{bmatrix} a+ug & b+uh & c+ui \\ d+vg & e+vh & f+vi \\ g & h & i \end{bmatrix}.$$

Indeed, u and v can be selected from the pair $\{0,1\}$. Apply case 1 to this matrix.

But the equations $ae-bd=dh-eg=ah-bg=0$ cannot all hold, because the first 2 columns of M are linearly independent. Thus cases 1, 2 and 3 cover all possibilities. ∎

Proposition 1 can be immediately applied to projective planes F^3/F.

PROPOSITION 2. *Let Δ be any triangle in the projective plane F^3/F. Then any $T\in \mathrm{PGL}(3,F)$ equals the product of 4 or fewer central collineations with axes (poles) lying on sides (vertices) of Δ.*

Proof. We give the proof for axes. The proof for poles is found by dualizing. Let A, B, C be the vertices of Δ, and let D be a point not on any side of Δ. Let $\{A,B,C,D\}$ be the system of reference points for F^3/F. Then

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

are respectively the points A,B,C,D. Let M be a matrix implementing T. Then $M\in \mathrm{GL}(3,F)$, and we have $M=N_1N_2N_3N_4$, where the N_i are the central matrices given in Proposition 1. The factors are the central collineations implemented by the N_i. The rest is clear. ∎

PROPOSITION 3. *Let Δ be any triangle in the projective plane F^3/F, and $T \in \mathrm{PGL}(3,F)$. Then T does not equal the product of 3 or fewer central collineations with distinct axes lying on the sides of Δ if and only if one of the following holds*:

(1) *T maps each vertex of Δ to a point on the opposite side of Δ.*
(2) *T maps each side of Δ to a line on the opposite vertex of Δ.*
(3) *T interchanges two sides of Δ.*

Proof. Let A, B, C, D and M be as in the proof of Proposition 2. Say

$$M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \in \mathrm{GL}(3,F).$$

It follows from Lemma 10 that T cannot be expressed as such a product if and only if all of the following 3 properties hold: [either $ae = bd$ or $a = e = 0$] and [either $ai = cg$ or $a = i = 0$] and [either $ei = fh$ or $e = i = 0$]. But $ae = bd$ means that T maps the line AB to a line on C, and $a = e = 0$ means that T maps A to a point on side BC and T maps B to a point on side AC. Similar observations about the other two properties show that T is not such a product if and only if for any two vertices P and Q of Δ, T either maps P and Q to points on the opposite sides, or maps side PQ to a line on the third vertex of Δ. But it follows (by inspection) that this last condition is equivalent to T satisfying one of (1), (2), (3). ∎

We dualize Proposition 3 to obtain

PROPOSITION 4. *Let Δ be any triangle in the projective plane F^3/F, and $T \in \mathrm{PGL}(3,F)$. Then T does not equal the product of 3 or fewer central collineations with distinct poles lying on the vertices of Δ if and only if one of the following holds*:

(1) *T maps each vertex of Δ to a point on the opposite side of Δ,*
(2) *T maps each side of Δ to a line on the opposite vertex of Δ,*
(3) *T interchanges two vertices of Δ.*

Thus it appears that "most" members of $\mathrm{PGL}(3,F)$ can be expressed as a product of 3 factors as described in Propositions 3 and 4. We conclude with an observation about the projective line F^2/F. We begin with

LEMMA 11. *Let $M \in \mathrm{GL}(2,F)$, and let $x_1, x_2, x_3, x_4 \in F$ such that $x_1x_2x_3x_4 = \det M$. Then there exist central matrices M_1, M_2, M_3, M_4 such that $M =$*

$M_1M_2M_3M_4$, $x_i = \det M_i$ *for* $i = 1,2,3,4$, *and each* M_i *is of one of the forms*

$$\begin{pmatrix} x_i & 0 \\ \vee & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & \vee \\ 0 & x_i \end{pmatrix}.$$

Proof. For some $u \in F$ the matrix

$$\begin{pmatrix} x_1^{-1} & 0 \\ u & 1 \end{pmatrix} M$$

has a nonzero entry in the 21th place. Call the left factor A_1, so that this product is A_1M. There is a $v \in F$ such that

$$\begin{pmatrix} 1 & v \\ 0 & x_2^{-1} \end{pmatrix} A_1 M$$

has x_3 in the 11th place. Call the left factor A_2, so that this product is A_2A_1M. Say

$$A_2A_1M = \begin{pmatrix} x_3 & a \\ b & c \end{pmatrix} = \begin{pmatrix} x_3 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} 1 & x_3^{-1}a \\ 0 & c - x_3^{-1}ab \end{pmatrix}.$$

Call the matrices on the right A_3 and A_4, so that $A_2A_1M = A_3A_4$. So $M = A_1^{-1}A_2^{-1}A_3A_4$, and $\det M = x_1x_2x_3x_4 = x_1x_2x_3(c - x_3^{-1}ab)$. Hence $x_4 = c - x_3^{-1}ab$. ∎

PROPOSITION 5. *Let* $|F| > 2$, *let* $T \in \mathrm{PGL}(2,F)$, *and let* A, B *be distinct points on the line* F^2/F. *Let* S_1, S_2, S_3, S_4 *be central collineations in* $\mathrm{PGL}(2,F)$. *Then* $S_1S_2S_3S_4T^{-1} \in \mathrm{PSL}(2,F)$ *if and only if there exist central collineations* G_1, G_2, G_3, G_4 *such that* $T = G_1G_2G_3G_4$, *such that* G_i *is conjugate to* S_i *in* $\mathrm{PGL}(2,F)$ *for* $i = 1,2,3,4$, *and such that each* G_i *leaves fixed either* A *or* B *(or both).*

The proof employs Lemma 11 the same way we employed Lemma 8 in the proof of Theorem 1. The proofs are so much alike that we leave this one to the reader.

Fewer than 4 factors will not in general suffice. Consider $M = \mathrm{diag}(\frac{1}{2}, 2)$ and $x_i = 1$, and compute directly.

4. CONCLUSION

We see now that Lemma 8 is our crucial result that proves Theorems 1, 2, 3, 4 and most of the rest of our work. We mention another corollary of Lemma 8 that is a detail. For any $M\in \mathrm{GL}(n,F)\setminus \mathrm{SL}(n,F)$, it is easy to see that $\det M$ has an nth root in F if and only if M is the product of n pairwise similar matrices in $\mathrm{GL}(n,F)$. Likewise any $T\in \mathrm{PGL}(n,F)$ lies in $\mathrm{PSL}(n,F)$ if and only if T equals the product of n pairwise conjugate members of $\mathrm{PGL}(n,F)$, provided either $n>2$ or $|F|>2$.

The factors given in Lemma 8 are not unique: Multiply on the left by $E'(1,u)^{-1}$ and on the right by $E'(1,u)$. But the product in Lemma 8 does determine uniquely the factors. Say $A_1\cdots A_n=B_1\cdots B_n$, where A_i and B_i are both factors as in Lemma 8. Then $A_1^{-1}B_1=A_2\cdots A_nB_n^{-1}\cdots B_2^{-1}$, and it follows from the form of the matrices that $A_1^{-1}B_1=I_n$ and $A_1=B_1$. Then $A_2\cdots A_n=B_2\cdots B_n$, and we argue similarly that $A_2=B_2$, and so forth. Finally, $A_i=B_i$ for all $i=1,\dots,n$.

Lemma 8 tells us, among other things, that any $M\in \mathrm{GL}(n,F)$ equals the product of n factors in $\mathrm{GL}(n,F)$, $M=M_1\cdots M_n$, where each M_i is either a central matrix or I_n. This is true even when M is a scalar matrix.

From Lemma 9, we saw that any $M\in \mathrm{GL}(n,F)$ with $\det M=\pm 1$ is the product of $n+2$ or fewer simple involutions. If F contains no root of the equation $x^n=(-1)^n$ other than 1 and -1, then $n+1$ or fewer factors suffice. Note that case (d) is trivial for any such field. Any ordered field is an example of such a field.

An analogue of Proposition 5 can be constructed in which $T\in \mathrm{PGL}(n,F)$ equals the product of $3n-2$ such central collineations. This was omitted to save space; the argument adds little to the proof of Proposition 5.

We conclude with several questions that could be the topic of further study.

1. Commutativity of multiplication in F was essential. Can some analogue of Lemma 8 or our other work be constructed when F is a skew field?

2. Does Lemma 8, or some modification of it, hold when "unitarily equivalent" replaces "similar" and F is the complex field? We can ask the same question when "orthogonally equivalent" replaces "similar" and F is the real field, or when "orthogonally equivalent" replaces "similar" and a bilinear form is imposed on the vector space F^n as in [7].

3. Let F be the complex field. Which $M\in \mathrm{GL}(n,F)$ has this property: for any $x_i\in F$ with $x_1\cdots x_n=\det M$, there exist normal central matrices M_i such that $M=M_1\cdots M_n$ and $x_i=\det M_i$ for $i=1,\dots,n$? One gathers from [5, Theorem 10] that some M do not have this property.

4. Same question as 3 where F is the real field.

The following questions are of special interest to geometers.

5. Proposition 2 holds for any Pappian projective plane. Does it hold for some more general kind of projective plane? Same question for Proposition 3.

6. Can synthetic proofs be given for Propositions 2, 3, 4 that do not involve matrix manipulation?

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